University of Nottingham
UK | CHINA | MALAYSIA

Thermodynamics and Fluid Mechanics 2

Fluids Topic 4: Dimensional analysis

Mirco Magnini

Fluids – Spring semester

Teaching week	w/b	Lectures (Physics B1, Mon h9-11)	Seminar session (Physics B1, Wed h9-10)
20	30 Jan	T4: Dimensional analysis	T4
21	06 Feb	T4: Dimensional analysis	T4
26	13 Mar	T5: Turbomachinery	T5
27	20 Mar	T5: Turbomachinery	T5
33	01 May	T6: Compressible flows (Tue 2nd: 12-1 Keighton Auditorium, Wed 3rd: 9-10 Physics B1)	T6 (Fri 5th: 12-1 Pope C16)
34	08 May	T6: Compressible flows (Wed 10th: 9-10 Physics B1, Wed 10th: 10-11 Pope C17)	T6 (Fri 12th: 12-1 Pope C16)
35	15 May	Revision (regular slots)	

Drag lab

- In person: L3 B10 faculty fluids lab
- Duration: from 31/01 to 14/03 (Tuesdays). The lab lasts for 2 hours.
- Coursework: drag lab report (15%)
- Deadline for submission: 2 weeks after lab
- Cannot attend? Ask me for sample data, deadline stays the same
- Want to extend deadline? Submit an EC

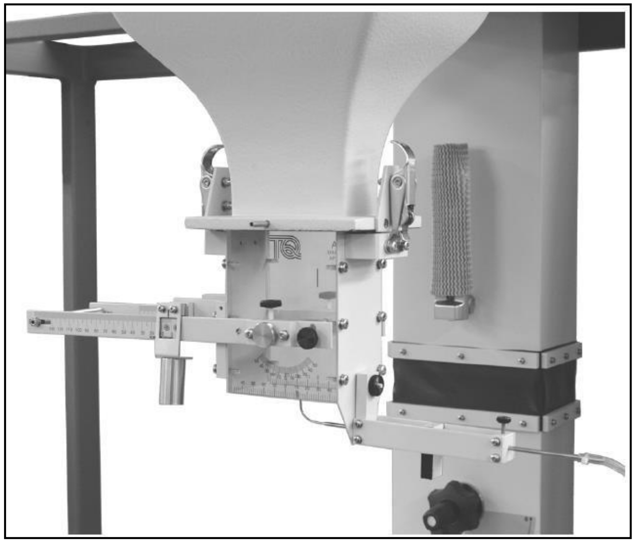


Image of drag lab apparatus

Drag lab: how does it work?

- Check out the Drag Lab Moodle tab
- Before the lab: Read the explanatory lab handout so you know what to expect
- During the lab: perform the experiments and write down data on the handout. BRING YOUR OWN COPY!
- After your lab takes place, process the data to calculate drag coefficients, Re numbers, and other parameters.
- Write up the report following the provided template.
- Submit it on Moodle, submission box under the drag lab tab.

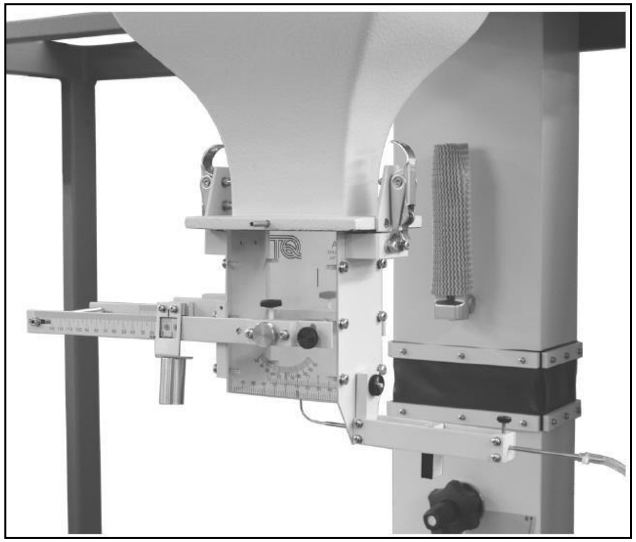


Image of drag lab apparatus

Why/where is it important?

- Most practical fluid flow problems are too complex to be solved analytically, and must be tested using experiments and/or CFD simulations
- In lab tests, it is not always possible to use the actual scale of the prototype, or the actual flow speed, or the actual fluid, and thus we use a model. How do we make sure that the model represents well the prototype?

Panasonic Toyota F1 Car and 50% Scale Wind Tunnel Model.

http://www.aero-performance-engineering.co.uk/wind-tunnel-testing

NASA wind tunnel with the scale model of an airplane

https://en.wikipedia.org/wiki/Wind_tunnel#/media/File:MD-11_12ft_Wind_Tunnel_Test.jpg

Topic 4 – Dimensional analysis

Why/where is it important?

- A fluid flow problem may depend on many different variables, can we group them in a smaller number of parameters?

Boundary Layer Wind Tunnel test platform
http://www.ccea.zju.edu.cn/cceaenglish/2016/0324/c6034a426975/page.htm

Ship model test towing tank
http://www.polandatsea.com/ship-model-towing-tank-opened-at-gdansk-university-of-technology/

The answer is **YES**! This can be done by applying the concepts of *physical similarity* and making use of *dimensional analysis* to extract *nondimensional parameters*.

Topic 4 – Dimensional analysis

Topic 4 can be studied in F. White, Ch. 5

- Introduction to physical similarity
- Dimensions and units
- Use of dimensional analysis
- The Buckingham (or Pi) theorem
- Nondimensionalisation of the mass/momentum equations
- Standard nondimensional groups in fluid mechanics
- Similarity and model testing
- Consequences of incomplete similarity

Further reading/assessment:

- F. White book, Ch. 5 and examples therein; problems in Ch. 5.
- Notes, exercises in Moodle.
- B. Massey, Mechanics of Fluids, Ch. 5.

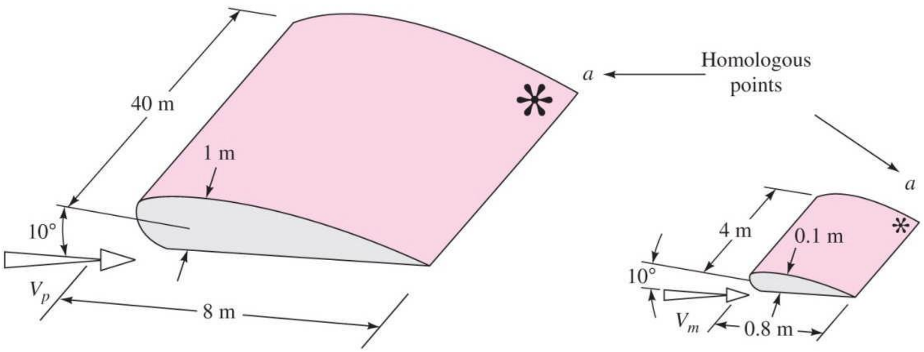
Topic 4 – Dimensional analysis

Learning outcomes:

- Be competent in your use of units and dimensions
- Be able to state what the main uses of dimensional analysis are
- Be able to state what non-dimensional groups are and why they are useful
- Be able to perform calculations using non-dimensional groups
- Be able to apply the Buckingham-Pi analysis where the variables involved are given
- Be able to discuss intelligently which variables might contribute in an experiment for straight forward cases
- Understand how to deal with variables that are themselves dimensionless
- Recognise common non-dimensional groups
- Be able to explain and apply dimensional similarity for model testing: geometric, kinematic and dynamic
- Be able to explain why similarity may not be possible and what the consequences of this (extrapolation, compromise) might be

Introduction to physical similarity

We want to characterise an airfoil (the prototype) for an airplane, in terms of drag and lift coefficients, by running experiments on a 1:10 scale model. How do me make sure that the prototype and its scaled model are physically similar?



- **Geometrical similarity**: all body dimensions in all three coordinates have the same linear scale ratio.
- **Kinematic similarity**: has to do with similarity of fluid motion, requires that velocities at corresponding points in the two flows are in the same direction and related by a constant scale factor in magnitude.
- **Dynamic similarity**: has to do with similarity of forces, and requires that the magnitude ratio of any two forces in one system must be the same as the magnitude ratio of the corresponding forces in the other system.

Kinematic and dynamic similarity are ensured by the equality of the governing nondimensional parameters.

Dimensions and units

In fluid mechanics, there are 4 basic **dimensions**:

- Mass, M (sometimes replaced by force, F)
- Length, L
- Time, T
- Temperature, θ

All other quantities can be found as combinations of these fundamental dimensions, e.g. velocity has dimensions L/T (or LT^{-1}).

Note that the **units** of velocity may vary depending on the system used, e.g. m/s, ft/s, km/h, but the dimension is always L/T.

Dimensions and units

Dimensions of fluid mechanics properties (from F. White)

Quantity	Symbol	Dimensions $MLT\Theta$	Dimensions $FLT\Theta$
Length	L	L	L
Area	A	L^2	L^2
Volume	$\mathcal{V}$	L^3	L^3
Velocity	V	LT^{-1}	LT^{-1}
Acceleration	dV/dt	LT^{-2}	LT^{-2}
Speed of sound	a	LT^{-1}	LT^{-1}
Volume flow	Q	L^3T^{-1}	L^3T^{-1}
Mass flow	$\dot{m}$	MT^{-1}	FTL^{-1}
Pressure, stress	p, σ, τ	$ML^{-1}T^{-2}$	FL^{-2}
Strain rate	$\dot{\epsilon}$	T^{-1}	T^{-1}
Angle	θ	None	None
Angular velocity	ω, Ω	T^{-1}	T^{-1}
Viscosity	μ	$ML^{-1}T^{-1}$	FTL^{-2}
Kinematic viscosity	ν	L^2T^{-1}	L^2T^{-1}
Surface tension	Υ	MT^{-2}	FL^{-1}
Force	F	MLT^{-2}	F
Moment, torque	M	ML^2T^{-2}	FL
Power	P	ML^2T^{-3}	FLT^{-1}
Work, energy	W, E	ML^2T^{-2}	FL
Density	ρ	ML^{-3}	FT^2L^{-4}
Temperature	T	Θ	Θ
Specific heat	c_p, c_v	$L^2T^{-2}\Theta^{-1}$	$L^2T^{-2}\Theta^{-1}$
Specific weight	γ	$ML^{-2}T^{-2}$	FL^{-3}
Thermal conductivity	k	$MLT^{-3}\Theta^{-1}$	$FT^{-1}\Theta^{-1}$
Thermal expansion coefficient	β	Θ^{-1}	Θ^{-1}

Dimensions and units

A nondimensional (dimensionless) group does not have any dimension or units.

For example, the Reynolds number:

$$Re = \frac{\rho U L}{\mu}$$

Quantity	SI unit	Dimensions
ρ, density	kg/m^3	ML^{-3}
v, velocity	m/s	LT^{-1}
D, length or diameter	m	L
μ, viscosity	kg/ms	$ML^{-1}T^{-1}$

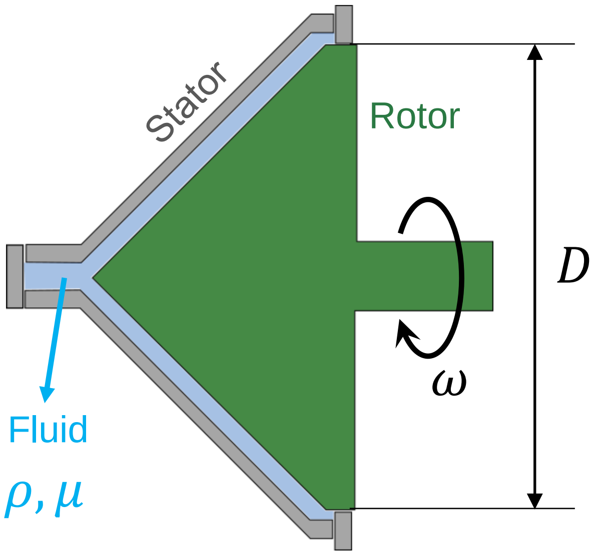
$$[Re] = \frac{[ML^{-3}][LT^{-1}][L]}{[ML^{-1}T^{-1}]} = [\,]$$

Use of dimensional analysis

Benefits deriving from appropriate use of dimensional analysis:

- Experiments (or simulations) can be performed on a scaled model
- Experiments (or simulations) can be run using different fluids
- Reduced number of important parameters, and therefore of tests
- Give generality to the results

Example: we want to study the torque on an enclosed rotating cone



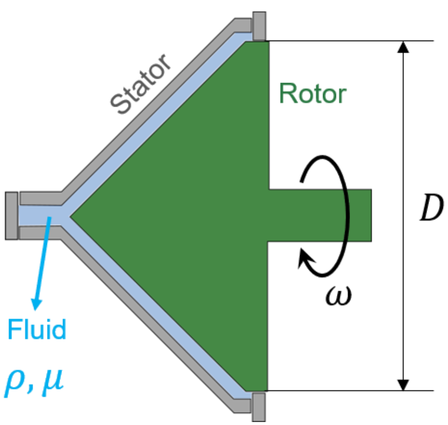
In order to maintain a constant ω, a certain torque T is required to overcome the shear stress exerted by the fluid on the surface of the rotating cone. Our experiment (or simulation) wants to answer the question *how are T and ω related?*

Use of dimensional analysis

How are T and ω related?

How many tests should we run?

- It depends on how many variables impact the flow.
- Here it comes the first 'engineering' art: we need to identify what are the variables impacting the flow. There is <u>never</u> an exact answer. The more we know the flow problem, the better it is. Let's say: ω, D, ρ, μ, so that we assume that $T = f(\omega, D, \rho, \mu)$.
- Then, to explore well the parameter space, we should run a few tests by varying these 4 parameters one at a time. Say, 10 runs for each parameter? This makes $10 \times 10 \times 10 \times 10 = 10^4$ experiments: **TOO MANY!!**



- Dimensional analysis comes to help. It suggests that we need only 2 nondimensional groups to relate ω, D, ρ, μ and T:

$$C_m = \frac{T}{\frac{1}{2}\rho\omega^2 D^5}, \quad Re = \frac{\rho\omega D^2}{\mu} \quad \Longrightarrow \boxed{C_m = g(Re)}$$

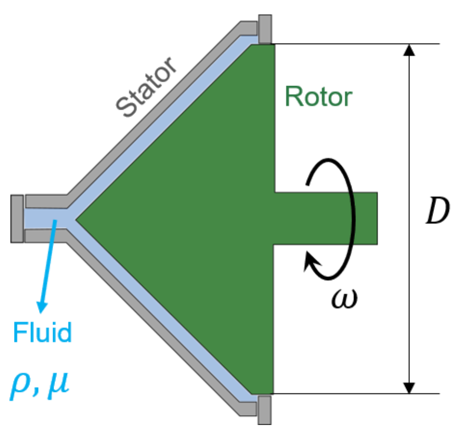
Use of dimensional analysis

How are T and ω related?

$$C_m = \frac{T}{\frac{1}{2}\rho\omega^2 D^5}, \quad Re = \frac{\rho\omega D^2}{\mu} \quad \Longrightarrow C_m = g(Re)$$

This result is very important: it suggests that the parameter space can be reduced to a single parameter, the rotating Reynolds number!

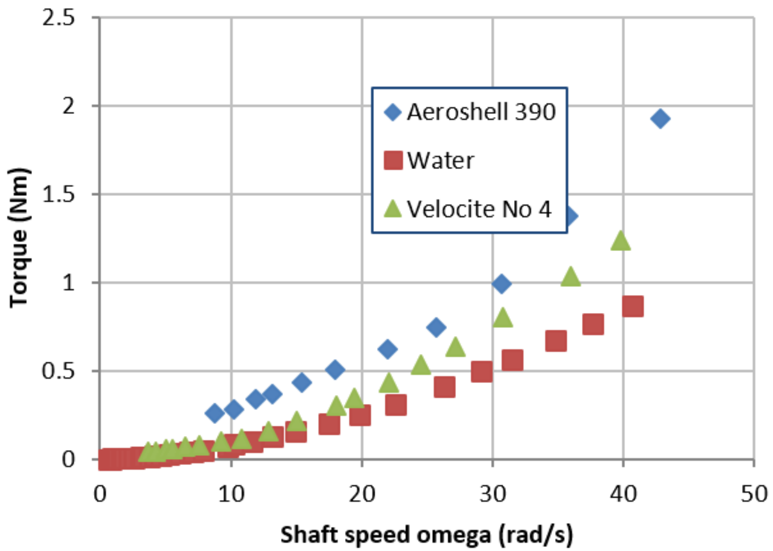
- Let's decide the series of experiments accordingly; I use 3 different fluids: water, Aeroshell oil 390 and Mobil Velocite oil No 4, this will allow to test different ρ, μ (but with 3, rather than 10×10 combinations); we change ω to test different rotation speeds; we fix D to a single value, because experimentally it may be costly to test many.

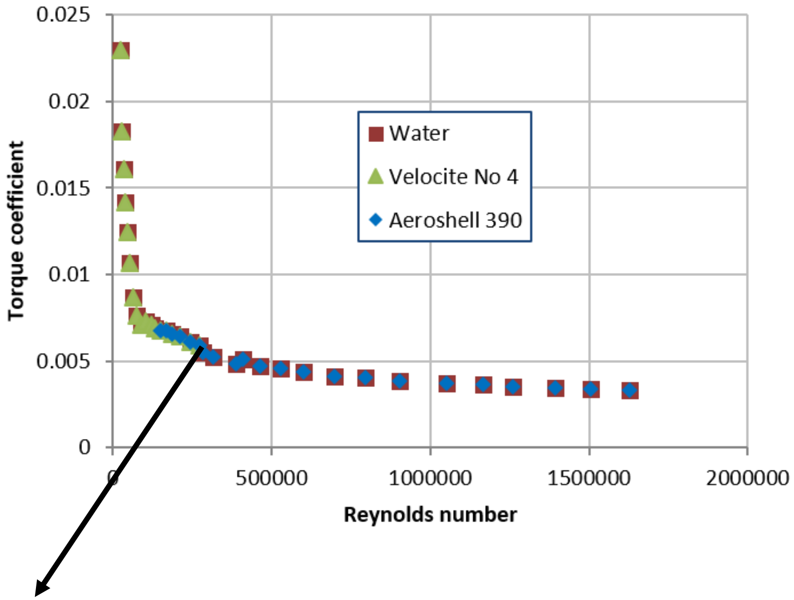


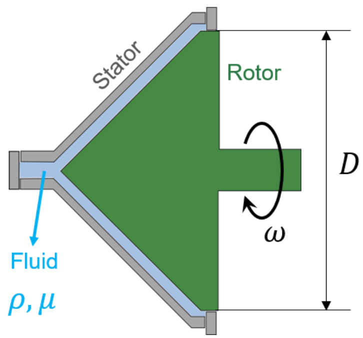
Use of dimensional analysis

How are T and ω related?

- Is the outcome of the dimensional analysis, $C_m = g(Re)$, correct? Let's see the data in <u>dimensional format</u> and <u>nondimensional format</u>







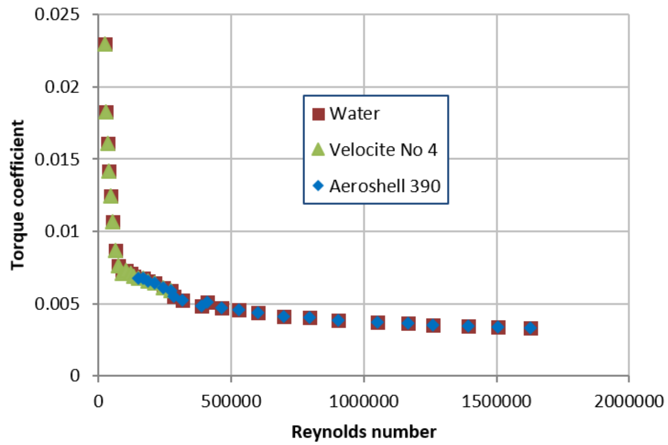
In nondimensional format, the data obtained with different fluids all collapse over a single curve $\Longrightarrow C_m = g(Re)$, the result of the dimensional analysis is verified!

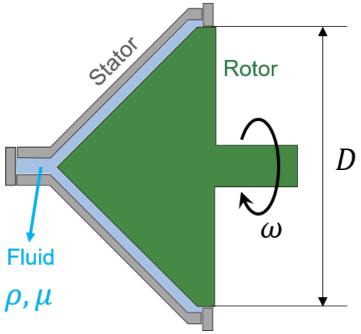
Use of dimensional analysis

How are T and ω related?

- The plot in nondimensional units is much more general than that in dimensional units: the curve of C_m vs Re is expected to be valid independently of the specific ω, D, ρ, μ, as long as C_m and Re are within the range of variation of the experiment.

- What if, once the data have been plotted in C_m vs Re format, the data points do not overlap? This would hint at some effects that have been left out of the dimensional analysis, for example:
  - ➢ Thickness of the gap between stator/rotor?
  - ➢ Cone angle?
  - ➢ Surface roughness?
  - ➢ Fluid heating due to viscous dissipation?





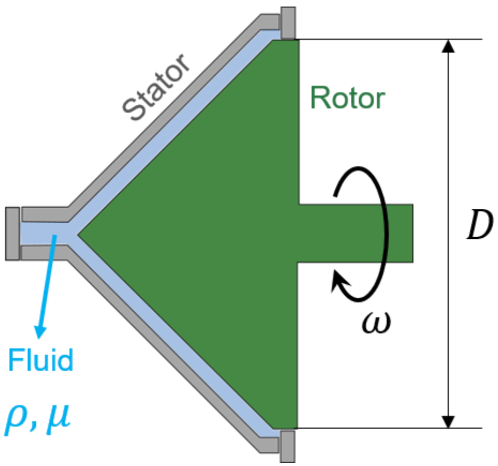
<u>Part of the art is deciding which are the important factors</u>

Worked example 1

The model of the rotating cone is usually applied to study windage power losses from rotating gears. Let's assume that in a real application, the operating fluid is air, the base diameter of the cone is $D = 12\ \mathrm{cm}$, and the cone will reach rotational speeds up to 3000 rpm. For health & safety reasons, in my lab I can't run the cone at such a high rotational speed, and therefore I want to use water as operating fluid. What should the rotational speed be, in order to obtain the same Reynolds number as in air?

Solution

Air: ρ=1.2 kg/m^3, μ =1.8·10^{-5} kg/(m·s). Water: ρ=1000 kg/m^3, μ=0.001 kg/(m·s).



$$Re_{air} = \frac{\rho_a \omega_a D^2}{\mu_a} = Re_{water} = \frac{\rho_w \omega_w D^2}{\mu_w}$$

$$\Longrightarrow \frac{\omega_w}{\omega_a} = \frac{\rho_a}{\rho_w}\frac{\mu_w}{\mu_a} = 0.0667$$

$$3000\ rpm \times 0.0667 = 200\ rpm$$

Some ‘self-evident truths’

- Dimensional homogeneity: If an equation truly expresses a proper relationship between variables in a physical process, it will be *dimensionally homogeneous*; that is, each of its additive terms will have the same dimensions.

Example: Bernoulli equation

$$p + \frac{1}{2}\rho U^2 + \rho g z = const$$ All the terms have dimensions of [$ML^{-1}T^{-2}$].

- Pure constants, such as $\frac{1}{2}, \pi, e$, or the arguments of mathematical functions such as $\sin()$, $\log()$, $\exp()$, have no dimensions.

- Angles, e.g. $\pi/4$, 90^o, are dimensionless.

The Buckingham Pi Theorem

How do we reduce the number of variables into a smaller number of dimensionless groups? A popular method is the *Buckingham Pi Theorem*.

- If a physical process is fully described by n variables and k dimensions, then $m = n - k$ dimensionless groups $(\Pi_1, \Pi_2, \dots, \Pi_m)$ are sufficient to describe the process. **Example:** suppose we have $n = 5$ variables that fully determine our process such that $v_1 = f(v_2, v_3, v_4, v_5)$, and the total number of dimensions is $k = 3$, e.g. M, L, T. The theorem says that $m = 5 - 3 = 2$, i.e. two nondimensional groups should be sufficient to describe the process with a functional relationship $\Pi_1 = g(\Pi_2)$.

- Then, we pick up the two variables we are most interested in, say v_1, v_5, and use the other three to form the dimensionless groups. It is important that these three do not themselves form a dimensionless group. Π_1, Π_2 are then formed by the variable we chose and power products of the three others:

$$\Pi_1 = v_1(v_2)^a(v_3)^b(v_4)^c, \qquad \Pi_2 = v_5(v_2)^d(v_3)^e(v_4)^f$$

The Buckingham Pi Theorem

$$\Pi_1 = v_1(v_2)^a(v_3)^b(v_4)^c, \qquad \Pi_2 = v_5(v_2)^d(v_3)^e(v_4)^f$$

- The exponents are obtained by the knowing that the groups are dimensionless:

$$[\Pi_1] = [v_1][v_2]^a[v_3]^b[v_4]^c = [M^0L^0T^0]$$

$$[\Pi_2] = [v_5][v_2]^d[v_3]^e[v_4]^f = [M^0L^0T^0]$$

Worked example 2

The torque of the rotating cone considered before is related to the rotation speed, fluid properties and cone base diameter, $T = f(\omega, D, \rho, \mu)$. Identify the appropriate non-dimensional groups.

Solution

- First find the dimensions of each variable:

Quantity	SI unit	Dimensions
ρ, density	kg/m^3	ML^{-3}
ω, angular velocity	rad/s	T^{-1}
D, length or diameter	m	L
μ, viscosity	kg/ms	$ML^{-1}T^{-1}$
T, torque	Nm	ML^2T^{-2}

  so we know that $k = 3$. $n = 5$, and therefore there are $5 - 3 = 2$ nondimensional groups.

- We need to choose 2 variables that are parameters of interest; for sure T, as it represents the dependent parameter; the other seems arbitrary at this stage, but experience suggests, for viscous flows, to take μ. T, μ: non-repeating variables

Worked example 2

- The unchosen variables are ω, D, ρ: repeating variables. This was a good choice: there is no way to combine them to form a dimensionless group (feel free to try).
- Therefore:

$$\Pi_1 = T(\rho)^a(D)^b(\omega)^c, \qquad \Pi_2 = \mu(\rho)^d(D)^e(\omega)^f$$

- To find the exponents, we use:

$$[\Pi_1] = [T][\rho]^a[D]^b[\omega]^c = [M^0L^0T^0]$$

$$\Longrightarrow [\Pi_1] = [ML^2T^{-2}][ML^{-3}]^a[L]^b[T^{-1}]^c = [M^0L^0T^0]$$

$$\Longrightarrow [\Pi_1] = [M^{1+a}\,L^{2-3a+b}\,T^{-2-c}] = [M^0L^0T^0]$$

$$\Longrightarrow M:\ 1 + a = 0 \Longrightarrow a = -1$$

$$\Longrightarrow L:\ 2 - 3a + b = 0 \Longrightarrow b = -5$$

$$\Longrightarrow T: -2 - c = 0 \Longrightarrow c = -2$$

$$\Pi_1 = \frac{T}{\rho\omega^2D^5}$$

Which is the momentum coefficient mentioned earlier, without the 1/2 constant at the denominator

Worked example 2

$$\Pi_1 = \frac{T}{\rho\omega^2 D^5}$$

Which is the moment coefficient mentioned earlier, without the 1/2 constant at the denominator

$$[\Pi_2] = [\mu][\rho]^d[D]^e[\omega]^f = [M^0L^0T^0]$$

$$\Rightarrow [\Pi_2] = [ML^{-1}T^{-1}][ML^{-3}]^d[L]^e[T^{-1}]^f = [M^0L^0T^0]$$

$$\Rightarrow [\Pi_2] = \left[M^{1+d}\, L^{-1-3d+e}\, T^{-1-f}\right] = [M^0L^0T^0]$$

$$\Rightarrow M\colon\ 1 + d = 0 \Rightarrow d = -1$$

$$\Rightarrow L\colon -1 - 3d + e = 0 \Rightarrow e = -2$$

$$\Rightarrow T\colon -1 - f = 0 \Rightarrow f = -1$$

$$\Pi_2 = \frac{\mu}{\rho\omega D^2}$$

Which is $1/Re$

Therefore, the two nondimensional groups are the momentum coefficient and the Reynolds number, with $C_M = f(Re)$.

Worked example 2b

Repeat Worked example 2, but this time using ρ and ω as non-repeating variables.

You will obtain two new nondimensional groups:

$$\Pi_1 = \frac{\rho T}{D\mu^2}, \qquad \Pi_2 = \frac{\omega D^3 \mu}{T}$$

- Although the procedure is correct, the functional relationship $\Pi_1 = f(\Pi_2)$ will not be useful, because the torque appears in both groups.
- It is the experience and the knowledge of fluid mechanics that will help us understand how to extract useful nondimensional groups.
- For example, when running experiments/simulations, the ‘useful’ nondimensional groups are those that make dimensional data collapse onto a single curve after nondimensionalisation, as seen in the rotating cone example.
- It is good when known dimensionless groups (e.g. the Re number) appear as a results of the analysis.

The Buckingham Pi Theorem

Summary of the procedure:

1. List and count the n variables involved in the problem. If important variables are missing, dimensional analysis will fail or yield incomplete insight.
2. List the dimensions of each variable and count the dimensions (M,L,T,θ) to establish k.
3. Decide the $m = n - k$ non-repeating and the k repeating variables:
   - The repeating variables should contain all the dimensions
   - The dependent variable (e.g. the torque) should be a non-repeating variable
   - Viscosity is usually a good non-repeating variable because it will bring up Re
   - Make sure that the repeating variables do not form a nondimensional group
4. Set up your pi-groups writing them down with the unknown coefficients.
5. Balance the exponents to find the form of each nondimensional group.
6. Verify, once the pi-groups are written, that they are actually dimensionless, to rule out possible calculation errors.

Worked example 3

The power input P to a centrifugal pump is a function of the volume flowrate Q, impeller diameter D, rotation rate ω, and fluid properties ρ and μ:

$$P = f(Q, \omega, D, \rho, \mu)$$

Rewrite this as a dimensionless relationship using ω, D, ρ as repeating variables.

Solution: see notes, under Worked example 13, page 18-19

$$\frac{P}{\rho \omega^3 D^5} = f\left(\frac{\mu}{\rho \omega D^2}, \frac{Q}{\omega D^3}\right)$$

Nondimensional groups and Navier-Stokes eqs

The nondimensional groups governing a specific fluid flow problem can be obtained also by nondimensionalisation of the mass/momentum equations.

For example, let's consider the mass/momentum equations governing the incompressible flow of a Newtonian fluid, in the absence of gravity effects:

$$\nabla \cdot \boldsymbol{V} = 0$$

$$\rho \left[\frac{\partial \boldsymbol{V}}{\partial t} + \boldsymbol{V} \cdot (\nabla \boldsymbol{V}) \right] = -\nabla p + \mu \nabla^2 \boldsymbol{V}$$

At any wall (of a pipe, or of a body immersed in the flow): $\boldsymbol{V} = 0$,

At the inlet and outlet: $\boldsymbol{V}, p$ are known.

These equations contain three basic dimensions, M, L, T; all the variables, $p, \boldsymbol{V}, x, y, z, t$ can be made dimensionless by using the density ρ and two other reference constants, for instance a reference velocity U and a reference length scale L. U can be the upstream velocity, L the diameter of the pipe or a characteristic length of the body immersed in the stream.

Nondimensional groups and Navier-Stokes eqs

Let's define dimensionless variables based on these reference scales:

$$\boldsymbol{V}^* = \frac{\boldsymbol{V}}{U} \qquad \nabla^* = L\nabla \qquad x^* = \frac{x}{L} \qquad y^* = \frac{y}{L} \qquad z^* = \frac{z}{L} \qquad t^* = t\frac{U}{L} \qquad p^* = \frac{p}{\rho U^2}$$

The asterisk identifies dimensionless variables. We can use these to replace the dimensional variables in the mass/momentum equations:

$$\nabla \cdot \boldsymbol{V} = 0 \Longrightarrow \frac{1}{L}\nabla^* \cdot (U\boldsymbol{V}^*) = 0 \Longrightarrow \boxed{\nabla^* \cdot \boldsymbol{V}^* = 0}$$ Nondimensional mass equation

$$\rho\left[\frac{\partial \boldsymbol{V}}{\partial t} + \boldsymbol{V} \cdot (\nabla \boldsymbol{V})\right] = -\nabla p + \mu \nabla^2 \boldsymbol{V}$$

$$\Longrightarrow \rho\left[\frac{\partial (U\boldsymbol{V}^*)}{\partial\left(\frac{L}{U}t^*\right)} + (U\boldsymbol{V}^*) \cdot \left(\frac{1}{L}\nabla^*(U\boldsymbol{V}^*)\right)\right] = -\frac{1}{L}\nabla^*(\rho U^2 p^*) + \mu\frac{1}{L^2}\nabla^{*2}(U\boldsymbol{V}^*)$$

$$\Longrightarrow \frac{\rho U^2}{L}\left[\frac{\partial \boldsymbol{V}^*}{\partial t^*} + \boldsymbol{V}^* \cdot (\nabla^* \boldsymbol{V}^*)\right] = -\frac{\rho U^2}{L}\nabla^* p^* + \mu\frac{U}{L^2}\nabla^{*2}\boldsymbol{V}^*$$

Nondimensional groups and Navier-Stokes eqs

$$\Rightarrow \frac{\rho U^2}{L}\left[\frac{\partial \boldsymbol{V}^*}{\partial t^*} + \boldsymbol{V}^* \cdot (\nabla^* \boldsymbol{V}^*)\right] = -\frac{\rho U^2}{L}\nabla^* p^* + \mu\frac{U}{L^2}\nabla^{*2}\boldsymbol{V}^*$$

Scale of inertial force

Scale of viscous force

Now divide everything by $\frac{\rho U^2}{L}$:

1/Reynolds!

$$\Rightarrow \frac{\partial \boldsymbol{V}^*}{\partial t^*} + \boldsymbol{V}^* \cdot (\nabla^* \boldsymbol{V}^*) = -\nabla^* p^* + \frac{\mu}{\rho U L}\nabla^{*2}\boldsymbol{V}^*$$

$$Re = \frac{inertia\ force}{viscous\ force} = \frac{\rho U L}{\mu}$$

$$\Rightarrow \frac{\partial \boldsymbol{V}^*}{\partial t^*} + \boldsymbol{V}^* \cdot (\nabla^* \boldsymbol{V}^*) = -\nabla^* p^* + \frac{1}{Re}\nabla^{*2}\boldsymbol{V}^*$$

Nondimensional momentum equation

Nondimensional boundary conditions:

At any wall: $\boldsymbol{V}^* = 0$,

At the inlet and outlet: $\boldsymbol{V}^*$, p^* are known.

Nondimensional groups and Navier-Stokes eqs

Nondimensional Navier-Stokes equations for incompressible flow:

$$\nabla^* \cdot \boldsymbol{V}^* = 0$$

$$\frac{\partial \boldsymbol{V}^*}{\partial t^*} + \boldsymbol{V}^* \cdot (\nabla^* \boldsymbol{V}^*) = -\nabla^* p^* + \frac{1}{Re} \nabla^{*2} \boldsymbol{V}^*$$

With boundary conditions:

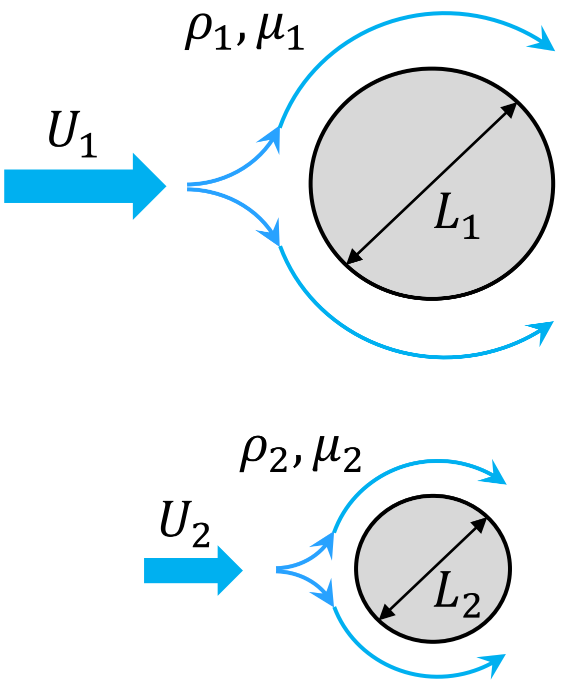
At any wall: $\boldsymbol{V}^* = 0$,

At the inlet and outlet: $\boldsymbol{V}^*$, p^* are known.

The nondimensional Navier-Stokes equations for an incompressible flow depend on one single parameter, Re. This means that $\boldsymbol{V}^*, p^* = \boldsymbol{V}^*, p^*(Re)$ only!

If we have two different configurations of (for example) flow past a cylinder, but where $Re_1 = Re_2$, then $\boldsymbol{V}_1^* = \boldsymbol{V}_2^*$ and $p_1^* = p_2^*$, which means that the velocity and pressure fields will be similar.

Example: flow past a cylinder



Standard nondimensional groups in fluids

In fluid mechanics, there are a few nondimensional groups that appear very often:

- Reynolds number:

$$Re = \frac{\rho U L}{\mu}$$

it represents the ratio of inertia forces over viscous forces, it is important in all viscous flows

- Froude number:

$$Fr = \frac{U^2}{gL}$$

it represents the ratio of inertia forces over gravity forces, and is important in flows with interfaces (e.g. gas-liquid)

- Weber number:

$$We = \frac{\rho U^2 L}{\sigma}$$

with σ being the surface tension coefficient, it represents the ratio of inertia to capillary forces, and is important in flows with strong surface tension effects, e.g. droplets, bubbles and jets

- Strouhal number:

$$St = \frac{fL}{U}$$

with f being a frequency, it is important in flows with velocity oscillations, for example in flow past a cylinder, $St \cong 0.21$ for a wide range of Reynolds numbers, $200 < Re < 10^5$

Standard nondimensional groups in fluids

In fluid mechanics, there are a few nondimensional groups that appear very often:

- Mach number:

$$Ma = \frac{U}{a}$$

with a being the speed of sound in the fluid, it gives an estimation of the importance of compressibility effects. When $Ma > 0.3$, the flow should be considered compressible; as a reference, air at 15 C and 1 atm has $a = 340\frac{m}{s} = 1224\ km/h$

There exist many others, as example the lift and drag coefficients, $C_{L,D} = \frac{L,D}{\frac{1}{2}\rho U^2 A}$ which compare the lift/drag force to the dynamic force of the flow, or the roughness ratio ϵ/L used for turbulent flows in a rough pipe or over a rough surface.

University of Nottingham
UK | CHINA | MALAYSIA

Standard nondimensional groups in fluids

F. White book provides an exhaustive list of dimensionless groups in fluid mechanics:

Parameter	Defiition	Qualitative ratio of effects	Importance
Reynolds number	$\mathrm{Re} = \frac{\rho U L}{\mu}$	$\frac{\text{Inertia}}{\text{Viscosity}}$	Almost always
Mach number	$\mathrm{Ma} = \frac{U}{a}$	$\frac{\text{Flow speed}}{\text{Sound speed}}$	Compressible flow
Froude number	$\mathrm{Fr} = \frac{U^2}{gL}$	$\frac{\text{Inertia}}{\text{Gravity}}$	Free-surface flow
Weber number	$\mathrm{We} = \frac{\rho U^2 L}{\Upsilon}$	$\frac{\text{Inertia}}{\text{Surface tension}}$	Free-surface flow
Rossby number	$\mathrm{Ro} = \frac{U}{\Omega_{\text{earth}} L}$	$\frac{\text{Flow velocity}}{\text{Coriolis effect}}$	Geophysical flows
Cavitation number (Euler number)	$\mathrm{Ca} = \frac{p - p_v}{\rho U^2}$	$\frac{\text{Pressure}}{\text{Inertia}}$	Cavitation
Prandtl number	$\mathrm{Pr} = \frac{\mu c_p}{k}$	$\frac{\text{Dissipation}}{\text{Conduction}}$	Heat convection
Eckert number	$\mathrm{Ec} = \frac{U^2}{c_p T_0}$	$\frac{\text{Kinetic energy}}{\text{Enthalpy}}$	Dissipation
Specific-heat ratio	$k = \frac{c_p}{c_v}$	$\frac{\text{Enthalpy}}{\text{Internal energy}}$	Compressible flow
Strouhal number	$\mathrm{St} = \frac{\omega L}{U}$	$\frac{\text{Oscillation}}{\text{Mean speed}}$	Oscillating flow
Roughness ratio	$\frac{\epsilon}{L}$	$\frac{\text{Wall roughness}}{\text{Body length}}$	Turbulent, rough walls
Grashof number	$\mathrm{Gr} = \frac{\beta \Delta T g L^3 \rho^2}{\mu^2}$	$\frac{\text{Buoyancy}}{\text{Viscosity}}$	Natural convection
Rayleigh number	$\mathrm{Ra} = \frac{\beta \Delta T g L^3 \rho^2 c_p}{\mu k}$	$\frac{\text{Buoyancy}}{\text{Viscosity}}$	Natural convection
Temperature ratio	$\frac{T_w}{T_0}$	$\frac{\text{Wall temperature}}{\text{Stream temperature}}$	Heat transfer
Pressure coefficient	$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho U^2}$	$\frac{\text{Static pressure}}{\text{Dynamic pressure}}$	Aerodynamics, hydrodynamics
Lift coefficient	$C_L = \frac{L}{\frac{1}{2}\rho U^2 A}$	$\frac{\text{Lift force}}{\text{Dynamic force}}$	Aerodynamics, hydrodynamics
Drag coefficient	$C_D = \frac{D}{\frac{1}{2}\rho U^2 A}$	$\frac{\text{Drag force}}{\text{Dynamic force}}$	Aerodynamics, hydrodynamics
Friction factor	$f = \frac{h_f}{(V^2/2g)(L/d)}$	$\frac{\text{Friction head loss}}{\text{Velocity head}}$	Pipe flow
Skin friction coefficient	$c_f = \frac{\tau_{\text{wall}}}{\rho V^2/2}$	$\frac{\text{Wall shear stress}}{\text{Dynamic pressure}}$	Boundary layer flow

Worked example 4

The pressure drop due to friction for flow in a long, smooth pipe is a function of average flow velocity, density, viscosity, pipe length and diameter $\Delta p = f(U, \rho, \mu, l, D)$. We want to know how Δp varies with U. Use the pi-theorem to rewrite this function in dimensionless form.

Solution

Step 1: List and count the n variables involved in the problem

$$\Delta p, U, \rho, \mu, L, D \Longrightarrow n = 6$$

Step 2: List the dimensions of each variable and count them to establish k.

$$[\rho] = [ML^{-3}]$$

$$[\mu] = [ML^{-1}T^{-1}]$$

$$[U] = [LT^{-1}] \qquad \Longrightarrow k = 3 \qquad \Longrightarrow m = n - k = 3$$ nondimensional groups

$$[D] = [L]$$

$$[l] = [L]$$

$$[\Delta p] = [ML^{-1}T^{-2}]$$

Worked example 4

Step 3: Decide the non-repeating and the repeating variables

- We need k=3 repeating variables and m=3 non-repeating variables
- Pressure is our dependent variable, so a good choice as a non-repeating variable
- We aim to express pressure as a function of velocity, so U should be another non-repeating variable
- Third non-repeating variable? We can imagine that the pipe length will be important in determining the pressure drop, so let's use l.
- The variables left as repeating ones are ρ, μ, D. Is this a good choice? Yes, they do not form a nondimensional group themselves, because there is no way to combine them to "balance" the time T at the denominator of μ.

Step 4: Set up your pi-groups writing them down with the unknown coefficients

$$\Pi_1 = \Delta p(\rho)^a(\mu)^b(D)^c, \qquad \Pi_2 = U(\rho)^d(\mu)^e(D)^f, \qquad \Pi_3 = l(\rho)^g(\mu)^h(D)^i$$

Worked example 4

Step 5: Balance the exponents to find the form of each nondimensional group

$$[\Pi_1] = [\Delta p][\rho]^a[\mu]^b[D]^c = [M^0L^0T^0]$$

$$\Longrightarrow [\Pi_1] = [ML^{-1}T^{-2}][ML^{-3}]^a[ML^{-1}T^{-1}]^b[L]^c = [M^0L^0T^0]$$

$$\Longrightarrow T: -2 - b = 0 \Longrightarrow b = -2$$

$$\Longrightarrow M: \ 1 + a + b = 0 \Longrightarrow a = 1$$

$$\Longrightarrow L: -1 - 3a - b + c = 0 \Longrightarrow c = 2$$

$$\Pi_1 = \frac{\rho D^2 \Delta p}{\mu^2}$$

$$[\Pi_2] = [U][\rho]^d[\mu]^e[D]^f = [M^0L^0T^0]$$

$$\Longrightarrow [\Pi_2] = [LT^{-1}][ML^{-3}]^d[ML^{-1}T^{-1}]^e[L]^f = [M^0L^0T^0]$$

$$\Longrightarrow T: -1 - e = 0 \Longrightarrow e = -1$$

$$\Longrightarrow M: d + e = 0 \Longrightarrow d = 1$$

$$\Longrightarrow L: 1 - 3d - e + f = 0 \Longrightarrow f = 1$$

$$\Pi_2 = \frac{\rho U D}{\mu} = Re$$

Worked example 4

$$[\Pi_3] = [l][\rho]^g[\mu]^h[D]^i = [M^0L^0T^0]$$

$$\Longrightarrow [\Pi_3] = [L][ML^{-3}]^g[ML^{-1}T^{-1}]^h[L]^i = [M^0L^0T^0]$$

$$\Longrightarrow T: -h = 0 \Longrightarrow h = 0$$

$$\Longrightarrow M: g + h = 0 \Longrightarrow g = 0$$

$$\Longrightarrow L: 1 - 3g - h + i = 0 \Longrightarrow i = -1$$

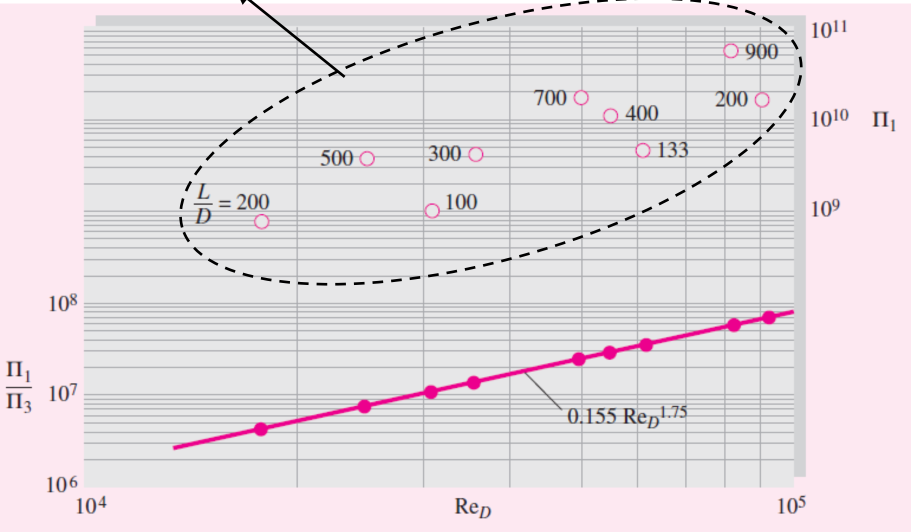
$$\Pi_3 = \frac{l}{D}$$

Therefore: $$\frac{\rho D^2 \Delta p}{\mu^2} = f\left(Re, \frac{l}{D}\right)$$

When plotting Π_1 vs Π_2, the points are scattered due to the effect of l/D

D, cm	L, m	Q, m³/h	Δp, Pa	ρ, kg/m³	μ, kg/(m · s)	V, m/s*
1.0	5.0	0.3	4,680	680†	2.92 E-4†	1.06
1.0	7.0	0.6	22,300	680†	2.92 E-4†	2.12
1.0	9.0	1.0	70,800	680†	2.92 E-4†	3.54
2.0	4.0	1.0	2,080	998‡	0.0010‡	0.88
2.0	6.0	2.0	10,500	998‡	0.0010‡	1.77
2.0	8.0	3.1	30,400	998‡	0.0010‡	2.74
3.0	3.0	0.5	540	13,550§	1.56 E-3§	0.20
3.0	4.0	1.0	2,480	13,550§	1.56 E-3§	0.39
3.0	5.0	1.7	9,600	13,550§	1.56 E-3§	0.67

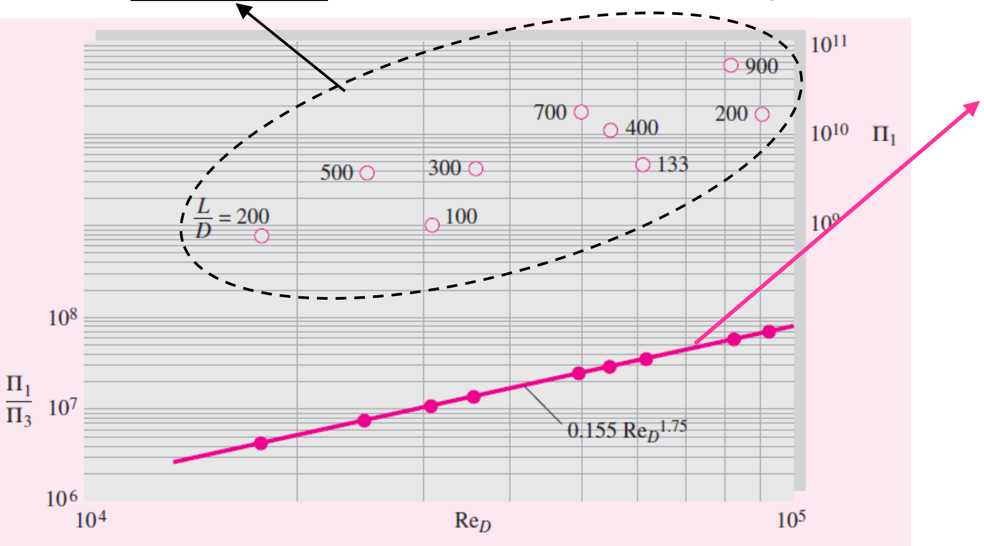
*$V = Q/A$, $A = \pi D^2/4$.
†Gasoline.
‡Water.
§Mercury.



(left) Pressure drop data in different conditions and (right) nondimensional plot, from F. White

Worked example 4

When plotting Π_1 vs Π_2, the points are scattered due to the effect of l/D



$$\frac{\Pi_1}{\Pi_3} = \frac{\rho D^3 \Delta p}{l\mu^2} = g(\Pi_2) = g(Re)$$

And, by fitting the experimental data:

$$\frac{\rho D^3 \Delta p}{l\mu^2} = 0.155 Re^{1.75}$$

Experience teaches us that, for a long pipe, Δp is proportional to the pipe length l; this allows us to rewrite our functional relationship as:

$$\frac{\rho D^2 \Delta p}{\mu^2} = f\left(Re, \frac{l}{D}\right) \Rightarrow \frac{\rho D^2 \Delta p}{\mu^2} = g(Re) \cdot \frac{l}{D} \qquad \Rightarrow \Pi_1 = g(\Pi_2) \cdot \Pi_3$$

Therefore, we can rearrange the relationship as: $\frac{\Pi_1}{\Pi_3} = g(\Pi_2)$

which means that the newly defined nondimensional group Π_1/Π_3 is a function of a single parameter. This makes all the points collapse onto a **single curve**.

Worked example 5

Consider the problem of flow past a smooth sphere. The free stream speed is U, the sphere diameter is D, the fluid density and viscosity are ρ, μ, and the drag force exerted on the sphere is F. Use the pi-theorem to prove that the drag coefficient can be expressed as a function of the Reynolds number only.

Solution

Step 1: List and count the n variables involved in the problem

$$\rho, \mu, U, D, F \quad \Longrightarrow n = 5$$

Step 2: List the dimensions of each variable and count them to establish k.

$$[\rho] = [ML^{-3}]$$

$$[\mu] = [ML^{-1}T^{-1}]$$

$$[U] = [LT^{-1}] \qquad \Longrightarrow k = 3 \quad \Longrightarrow m = n - k = 2 \quad \text{nondimensional groups}$$

$$[D] = [L]$$

$$[F] = [MLT^{-2}]$$

Worked example 5

Step 3: Decide the non-repeating and the repeating variables

- We need k=3 repeating variables and m=2 non-repeating variables
- Force is our dependent variable, so a good choice as a non-repeating variable
- Among the remaining ones, viscosity is usually another good candidate
- The variables left as repeating ones are ρ, U, D. Is this a good choice? Yes, they do not form a nondimensional group themselves, because there is no way to combine them to “balance” the time T at the denominator of U

Step 4: Set up your pi-groups writing them down with the unknown coefficients

$$\Pi_1 = F(\rho)^a(U)^b(D)^c, \qquad \Pi_2 = \mu(\rho)^d(U)^e(D)^f$$

Step 5: Balance the exponents to find the form of each nondimensional group

$$[\Pi_1] = [F][\rho]^a[U]^b[D]^c = [M^0L^0T^0]$$

$$\Longrightarrow [\Pi_1] = [MLT^{-2}][ML^{-3}]^a[LT^{-1}]^b[L]^c = [M^0L^0T^0]$$

Worked example 5

Step 5: Balance the exponents to find the form of each nondimensional group

$$[\Pi_1] = [F][\rho]^a[U]^b[D]^c = [M^0L^0T^0]$$

$$\Longrightarrow [\Pi_1] = [MLT^{-2}][ML^{-3}]^a[LT^{-1}]^b[L]^c = [M^0L^0T^0]$$

$$\Longrightarrow M:\ 1 + a = 0 \Longrightarrow a = -1$$

$$\Longrightarrow T: -2 - b = 0 \Longrightarrow b = -2$$

$$\Longrightarrow L: 1 - 3a + b + c = 0 \Longrightarrow c = -2$$

$$\Pi_1 = \frac{F}{\rho U^2 D^2}$$

Which is C_D, although we are used to see it as

$$C_D = \frac{F}{\frac{1}{2}\rho U^2 A}$$

$$[\Pi_2] = [\mu][\rho]^d[U]^e[D]^f = [M^0L^0T^0]$$

$$\Longrightarrow [\Pi_2] = [ML^{-1}T^{-1}][ML^{-3}]^d[LT^{-1}]^e[L]^f = [M^0L^0T^0]$$

$$\Longrightarrow M:\ 1 + d = 0 \Longrightarrow d = -1$$

$$\Longrightarrow T: -1 - e = 0 \Longrightarrow e = -1$$

$$\Longrightarrow L: -1 - 3d + e + f = 0 \Longrightarrow f = -1$$

$$\Pi_2 = \frac{\mu}{\rho U D}$$

Which is $1/Re$

Therefore, we have verified that: $C_D = f(Re)$.

Step 6: Work out their units to verify that Π_1, Π_2 are actually nondimensional.

Worked example 5b

Let's now consider that the sphere has a surface roughness ε.

Repeat the exercise to include this additional variable.

Solution

There is no need to repeat the analysis from scratch. We know already that:

$$\Pi_1 = \frac{F}{\rho U^2 D^2}, \qquad \Pi_2 = \frac{\mu}{\rho U D}$$

were sufficient for the $n = 5$ parameters ρ, μ, U, D, F. If we add one parameter, $n = 6$ while $k = 3$ stays the same, and therefore $m = n - k = 3 \Longrightarrow$ there will be a third nondimensional group that we have to find.

The most natural choice for the non-repeating variable to use is the new variable ε:

$$\Pi_3 = \varepsilon\,(\rho)^d (U)^e (D)^f$$

Therefore:

$$[\Pi_3] = [\varepsilon][\rho]^a [U]^b [D]^c = [M^0 L^0 T^0]$$

$$\Longrightarrow [\Pi_3] = [L][ML^{-3}]^a [LT^{-1}]^b [L]^c = [M^0 L^0 T^0]$$

Worked example 5b

$$[\Pi_3] = [\varepsilon][\rho]^a[U]^b[D]^c = [M^0L^0T^0]$$

$$\Rightarrow [\Pi_3] = [L][ML^{-3}]^a[LT^{-1}]^b[L]^c = [M^0L^0T^0]$$

$$\Rightarrow M:\ a = 0 \Rightarrow a = 0$$

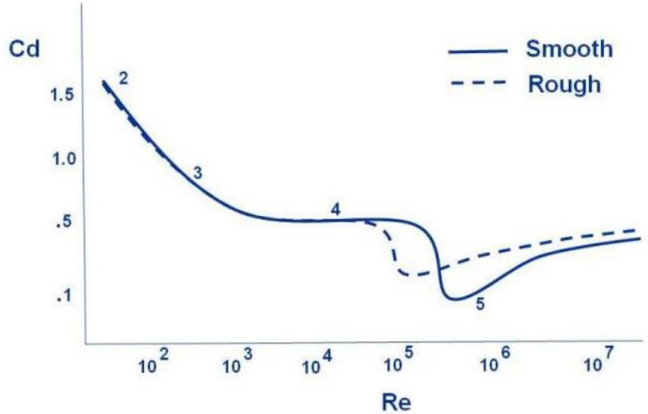
$$\Rightarrow T: -b = 0 \Rightarrow b = 0$$

$$\Rightarrow L: 1 - 3a + b + c = 0 \Rightarrow c = -1$$

$$\Pi_3 = \frac{\varepsilon}{D}$$

Which is the roughness parameter

Therefore, for flow over a rough sphere, we have: $C_D = f(Re, \varepsilon/D)$, i.e. the drag coefficient depends on both the Reynolds number and the roughness parameter, as we have seen in Lift&Drag.



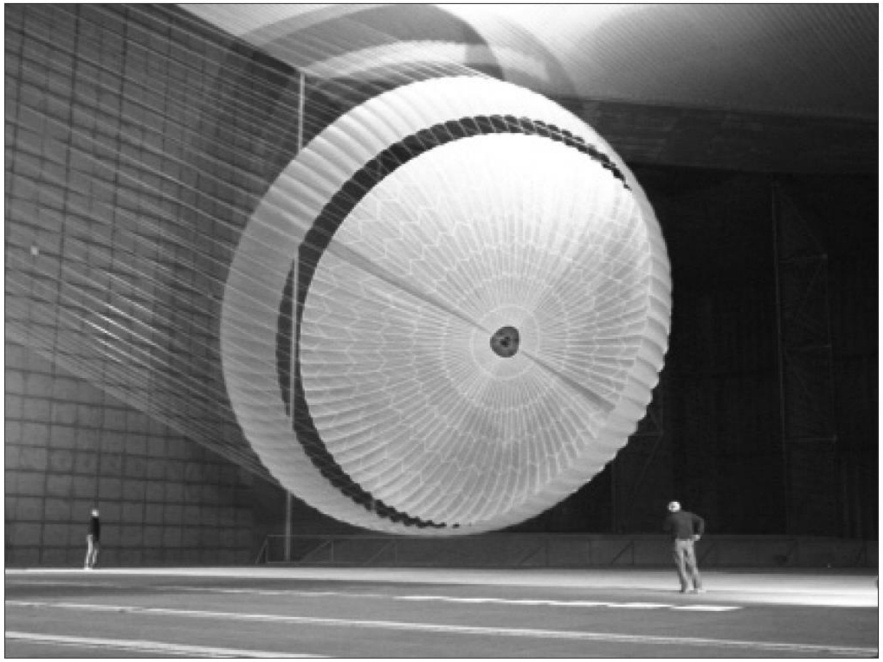
Worked example 6

The full-scale parachute in the figure below, d_1=16.5 m of diameter, has a drag force of D_1=4226 N when tested in a wind tunnel at a velocity of U_1=20 km/h in air at 20 ^oC and 1 atm. The scaled model of this parachute has a diameter of d_2=1.7 m. (a) For dynamic similarity, what should be the air velocity for the model, when tested in the same tunnel? (b) What is the expected drag force? (air: ρ=1.2 kg/m^3, μ =1.8·10^{-5} kg/(m·s))

Solution

The problem can be modeled as flow past a bluff body. We know from dimensional analysis (and from T3) that $C_D = f(Re)$, and therefore dynamic similarity is achieved when:

$$Re_1 = \frac{\rho U_1 d_1}{\mu} = Re_2 = \frac{\rho U_2 d_2}{\mu}$$



Full-scale test of a parachute for slowing down the descent of a roving laboratory on Mars, from F. White

Worked example 6

The problem can be modeled as flow past a bluff body. We know from dimensional analysis that $C_D = f(Re)$, and therefore dynamic similarity is achieved when:

$$Re_1 = \frac{\rho U_1 d_1}{\mu} = Re_2 = \frac{\rho U_2 d_2}{\mu}$$

and, since the same fluid (air) is used in both tests:

$$U_1 d_1 = U_2 d_2 \Longrightarrow U_2 = U_1 \frac{d_1}{d_2} = 194 \frac{km}{h} = 54 \frac{m}{s}$$ Which responds to (a)

For (b), we know that the drag force is expressed as:

$$D = \frac{1}{2} C_D \rho U^2 \frac{\pi d^2}{4}$$

And, because of dynamic similarity, C_D is the same for the two parachutes. We can work out C_D from the known value of D_1 and use it to calculate D_2, or recognise that:

$$D = \frac{1}{2} C_D \rho U^2 \frac{\pi d^2}{4} = \frac{\pi}{8} C_D \rho (U^2 d^2) = \frac{\pi}{8} C_D \rho \left(\frac{\mu^2 Re^2}{\rho^2} \right) = \frac{\pi}{8} C_D \frac{\mu^2 Re^2}{\rho}$$

Therefore, since C_D and Re are the same, $D_1 = D_2 = 4226\ N$: although the model is almost a 1:10 reproduction, <u>the drag force is the same</u>!

Similarity and model testing

- Nowadays, no aircraft, airfoil, vehicle, ship, turbomachinery, etc., are built before exhaustive tests are carried out on models in a wind-tunnel.
- *Similitude is the theory (and art) of predicting prototype perfomance from model observations*.
- Results taken from tests performed under one set of conditions are applied to another set of conditions.

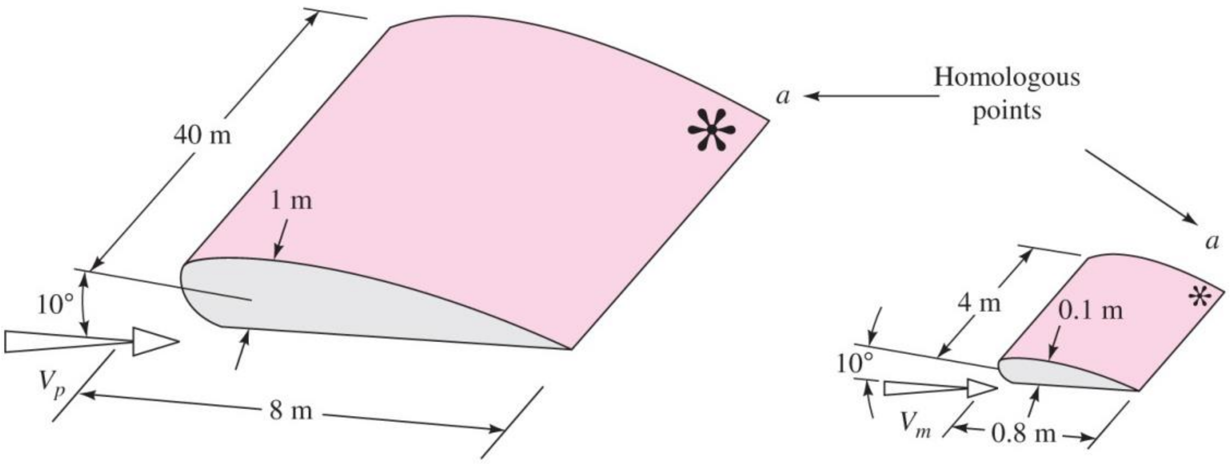
Panasonic Toyota F1 Car and 50% Scale Wind Tunnel Model.
http://www.aero-performance-engineering.co.uk/wind-tunnel-testing

- For any comparison between model and prototype to be valid, the sets of conditions associated with each must be *physically similar*.
- Two systems are physically similar in respect to certain specified physical quantities (length, velocity, force) when the ratio of corresponding magnitudes of these quantities between the two systems is everywhere the same.
- <u>Physical similarity is achieved when all relevant dimensionless groups have the same corresponding values for the model and the prototype</u>.

Geometric similarity

- Two systems are geometrically similar when the ratio of any length in one system to the corresponding length in the other system is everywhere the same.
- All angles must be preserved.
- All flow directions must be preserved.
- The orientation of model and prototype with respect to the surroundings must be identical.

1:43 model of F1 car



The prototype **The 1:10 model**

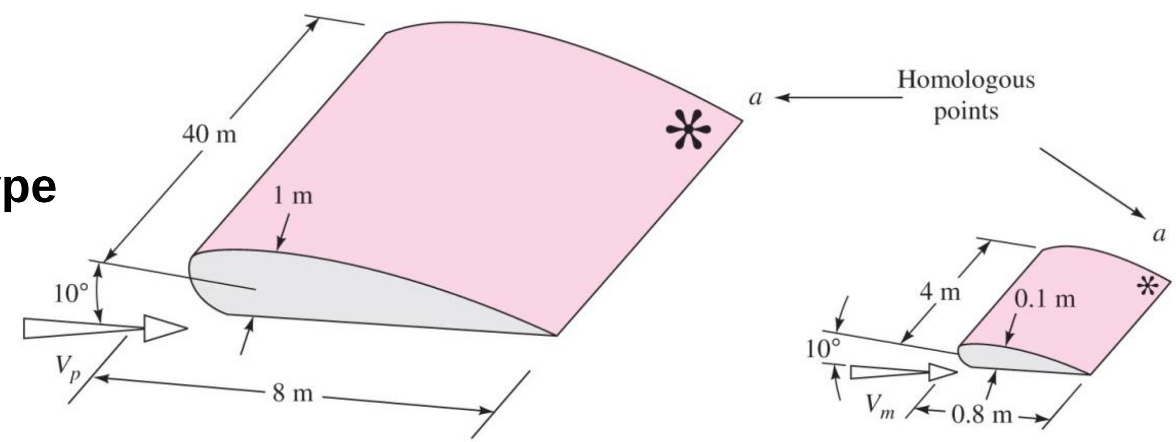
Geometric similarity

Example: for geometric similarity, the 1:10 model below must have

- Thickness, width and length 1/10 those of the prototype.
- Nose radius 1/10 that of the prototype.
- Surface roughness 1/10 that of the prototype.
- Any protruding fasteners (screws, etc) should be of the same scale, in the same homologous position and protrude only 1/10 of full size.
- Any coatings (e.g. paint) should be only 1/10 as thick.
- The angle of attack must be the same!

As you can imagine, some of these are not easy to achieve in practice!

The prototype



The 1:10 model

Kinematic similarity

Kinematic similarity is similarity of motion. Two systems have kinematic similarity when:

a) They are geometrically similar.

b) Velocities at corresponding (homologous) points in the two flows are in the same direction and are related by a constant scale factor in magnitude.

c) Flow regimes, e.g. laminar, turbulent, compressible, etc., must be the same.

Example: airfoil, model and prototype are geometrically similar, with $1{:}\,\alpha$ length ratio

- Homologous points: $x_m = x_p/\alpha \, , y_m = y_p/\alpha$
- u_m at (x_m, y_m) must have same direction as u_p at (x_p, y_p)
- $u_p/u_m = \beta$ is constant at homologous points

Prototype, subscript p

Model, subscript m

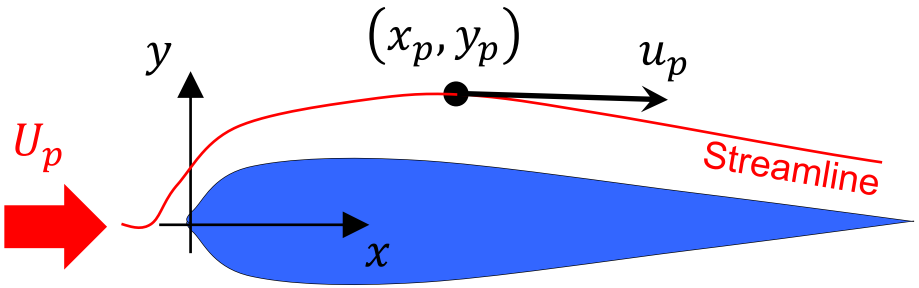
Kinematic similarity

Example: airfoil, model is geometrically similar, with $1\!:\!\alpha$ length ratio

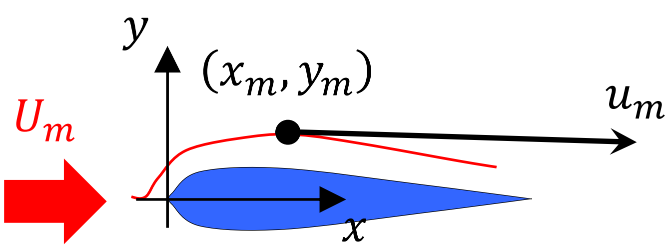
- Homologous point: $x_m = x_p/\alpha\,, y_m = y_p/\alpha$
- u_m at (x_m, y_m) has same direction as u_p at (x_p, y_p)
- $u_p/u_m = \beta$ is constant at homologous points

The value of β is defined by the equality of the governing nondimensional parameters, obtained via dimensional analysis. For instance, for a smooth airfoil and incompressible flow, kinematic and dynamic similarity are achieved if $Re_p = Re_m$. Therefore:

$$Re_p = \frac{\rho_p U_p L_p}{\mu_p} = Re_m = \frac{\rho_m U_m L_m}{\mu_m} \Longrightarrow \beta = \frac{U_p}{U_m} = \frac{L_m}{L_p}\frac{\rho_m}{\rho_p}\frac{\mu_p}{\mu_m}$$



Prototype, subscript p



Model, subscript m

Dynamic similarity

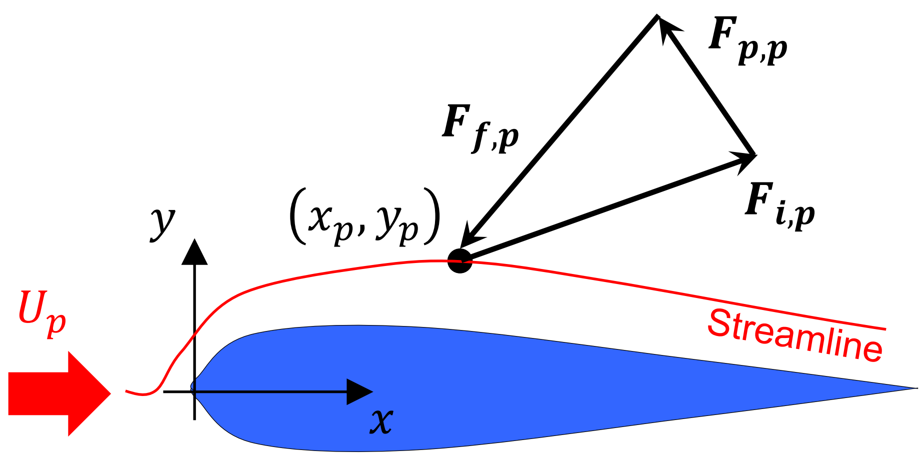
Dynamic similarity is similarity of forces. Two systems have dynamic similarity when:

a) They are geometrically similar.

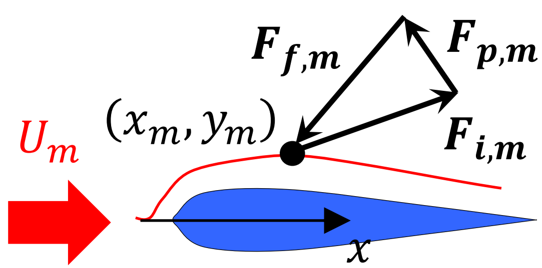
b) At corresponding (homologous) points in the two flows, identical kind of forces are parallel and are related by a constant scale factor.

Example: consider two homologous points around two geometrically similar airfoils

- Assume that there are 3 forces acting: inertia, $\boldsymbol{F_i}$, friction, $\boldsymbol{F_f}$, pressure, $\boldsymbol{F_p}$
- These forces must form a closed polygon: $\boldsymbol{F_i} = \boldsymbol{F_f} + \boldsymbol{F_p}$



Prototype, subscript p

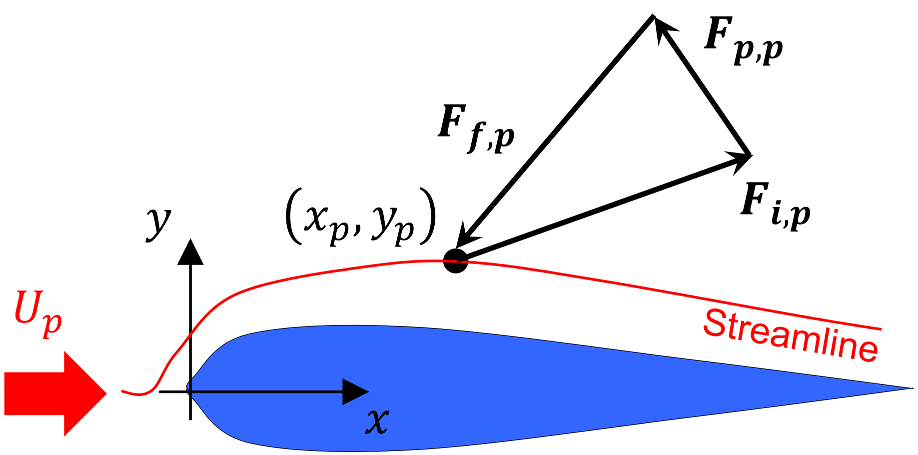


Model, subscript m

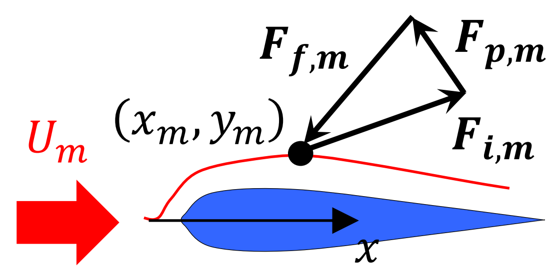
Dynamic similarity

Example: consider two homologous points around two geometrically similar airfoils

- Assume that there are 3 forces acting: inertia, $\boldsymbol{F_i}$, friction, $\boldsymbol{F_f}$, pressure, $\boldsymbol{F_p}$.
- These forces must form a closed polygon: $\boldsymbol{F_i} = \boldsymbol{F_f} + \boldsymbol{F_p}$.
- $\boldsymbol{F_{f,p}}$ and $\boldsymbol{F_{f,m}}$ must be parallel, $\boldsymbol{F_{i,p}}$ and $\boldsymbol{F_{i,m}}$ must be parallel, and so on.
- Force magnitude ratios must be related by a constant scale factor: $\dfrac{F_{i,p}}{F_{i,m}} = \dfrac{F_{f,p}}{F_{f,m}} = \dfrac{F_{p,p}}{F_{p,m}}$
- Therefore, the component forces have the same ratio of magnitude between the two flows, and thus the force polygons must be geometrically similar!



Prototype, subscript $\boldsymbol{p}$



Model, subscript $\boldsymbol{m}$

Dynamic similarity

How do we make sure that, at homologous points, $\frac{F_{i,p}}{F_{i,m}} = \frac{F_{f,p}}{F_{f,m}} = \frac{F_{p,p}}{F_{p,m}}$?

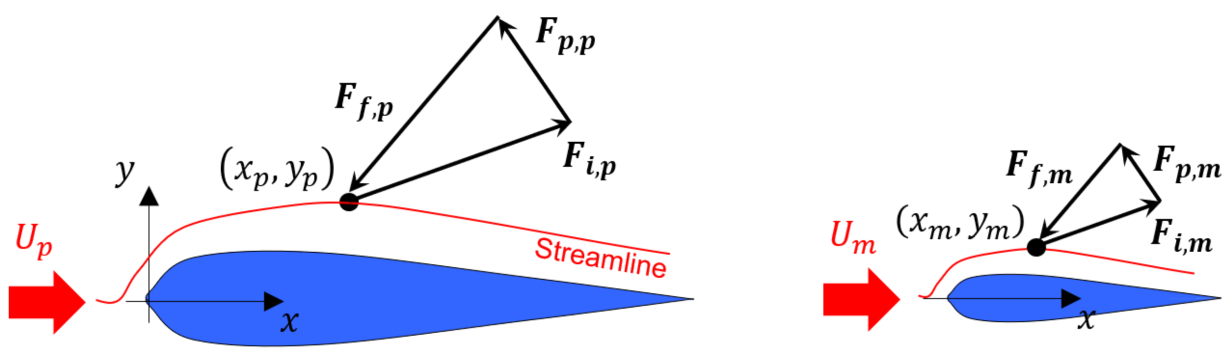
- Let's start with observing the equality of inertial and friction forces ratios:

$$\frac{F_{i,p}}{F_{i,m}} = \frac{F_{f,p}}{F_{f,m}}$$ And same for other kind of forces…

- Rearranging: $\frac{F_{i,p}}{F_{f,p}} = \frac{F_{i,m}}{F_{f,m}}$ ➡ At homologous points, the magnitude ratio of any two forces on one system must be the same as the magnitude ratio of the corresponding forces in the other system

- Estimation of F_i: $F_i = m\ a$
  - Mass: $m \sim \rho V \sim \rho L^3$
  - Acceleration: $a \sim \frac{v}{t} \sim \frac{U}{L/U}$

$$\Longrightarrow F_i \sim \rho L^3 \frac{U}{L/U} = \rho L^2 U^2$$



Prototype, subscript p **Model, subscript m**

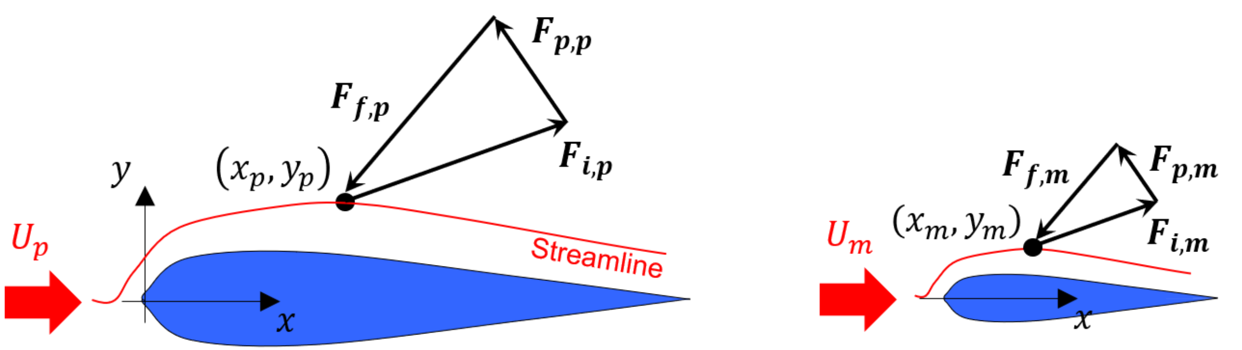
Dynamic similarity

- Estimation of F_i: $F_i = m\,a \quad \Longrightarrow F_i \sim \rho L^3 \dfrac{U}{L/U} = \rho L^2 U^2$

- Estimation of F_f: $F_f = \tau\, A \quad \Longrightarrow F_f \sim \mu \dfrac{U}{L} L^2 = \mu U L$
  - Area: $A \sim L^2$
  - Shear stress: $\tau \sim \mu \dfrac{\partial u_j}{\partial x_k} \sim \mu \dfrac{U}{L}$

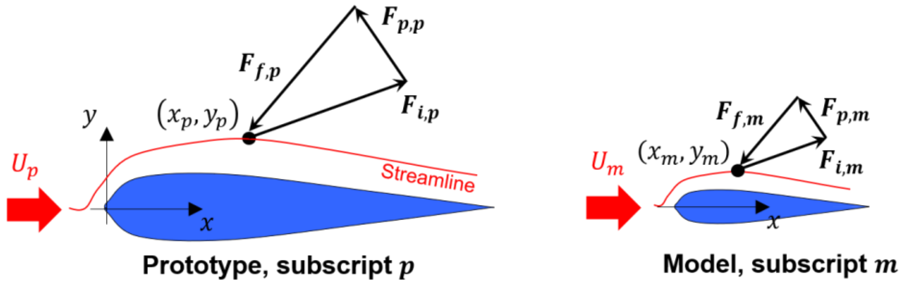
- Therefore: $\dfrac{F_i}{F_f} \sim \dfrac{\rho L^2 U^2}{\mu U L} = \dfrac{\rho U L}{\mu} = Re$

At homologous points, the magnitude ratio of inertia/viscosity in one system is the same as that in the other system **if the Reynolds number of the flow is the same!!**



Prototype, subscript p **Model, subscript m**

Dynamic similarity



Important points:

1. Dynamic similarity is achieved if the relevant nondimensional groups in prototype and model are the same.
2. Forces give rise to acceleration, thus velocities. Therefore, similarity of forces also produces similarity of flow patterns, i.e. kinematic similarity.

As a general rule, dynamic and kinematic similarity are ensured if:

- For compressible flow, prototype and model Reynolds and Mach number, and specific-heat ratio, are correspondingly equal.
- For incompressible flow with no free-surface, prototype and model Reynolds numbers are equal.
- For incompressible flow with a free-surface, Reynolds and Froude numbers are equal (may require equality of Weber and cavitation number as well, in some situations).

Worked example 7

In some practical cases, complete dynamic similarity is impossible to achieve. This example shows one such case.

A prototype boat is to be tested at a model scale of 1:50. This is an incompressible free surface flow and to ensure dynamic similarity both Reynolds number and Froude number must be equal. What kinematic viscosity must the model working fluid have if the prototype working fluid is water ($\nu=10^{-6}$ m²/s)?

1 cSt = 10^{-6} m²/s

Fluid	Temperature (°C)	Kinematic viscosity (cSt)
Distilled water	20	1.004
Olive oil	37.8	43.2
mercury	21.1	0.118
kerosene	20	2.71
Freon 11	21.1	0.21
Acetone	20	0.41

Solution

$$Re_p = \frac{U_p L_p}{\nu_p} = Re_m = \frac{U_m L_m}{\nu_m} \Longrightarrow \frac{\nu_m}{\nu_p} = \frac{U_m}{U_p}\frac{L_m}{L_p}$$

$$\frac{L_m}{L_p} = \frac{1}{50}$$

$$Fr_p = \frac{U_p^2}{gL_p} = Fr_m = \frac{U_m^2}{gL_m} \Longrightarrow \frac{U_m}{U_p} = \sqrt{\frac{L_m}{L_p}} = \sqrt{\frac{1}{50}}$$

$$\Longrightarrow \frac{\nu_m}{\nu_p} = \frac{U_m}{U_p}\frac{L_m}{L_p} = \sqrt{\frac{1}{50}}\cdot\frac{1}{50} = 0.00283$$

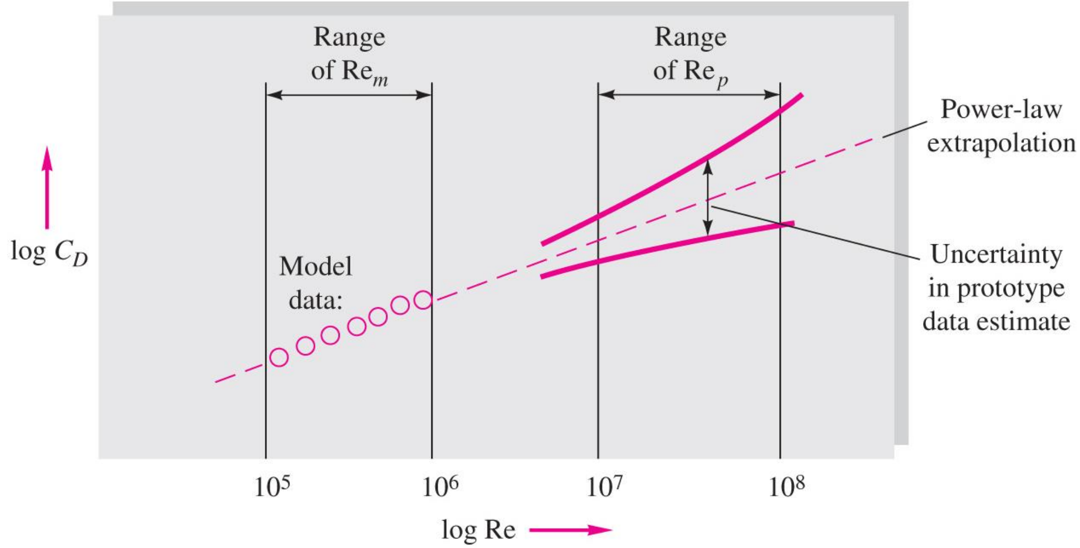
There is no fluid with kinematic viscosity 0.00283 smaller than that of water!

Consequences of incomplete similarity

Therefore, in this example it is not possible to achieve complete dynamic similarity.

How can we make sure that tests with the model represent well the prototype?

- In this specific case, tests are usually run with water, as it is a convenient fluid.
- This violates the Reynolds number similarity, but it does not matter too much because the Froude number is the dominant parameter in free-surface flows. Hence, we make sure that the Froude number is the same for both prototype and model.
- The Reynolds number of the model will be smaller. The low-Re model data will be extrapolated to the desired high-Re conditions.
- Extrapolation increases the uncertainty in the data prediction.
- It is responsibility of the engineer to judge the validity of the extrapolated data.



Topic 4 – Dimensional analysis

Now you should be able to:

- List units and dimensions of the usual variables appearing in fluid mechanics
- Explain the benefits deriving from appropriate use of dimensional analysis
- Apply the Buckingham-Pi theorem to derive governing nondimensional groups
- Recognise the usual nondimensional groups appearing in fluid mechanics
- Identify the requirements for geometric, kinematic and dynamic similarity
- Recognise that complete similarity is not always possible, and what the potential remedies are

Further reading/assessment:

- F. White book, Ch. 5 and examples therein; problems in Ch. 5.
- Notes/exercises in Moodle.
- B. Massey, Mechanics of Fluids, Ch. 5.

Seminar

Worked example 8

From 17/18 exam, long question Q15.

The drag force D developed on a boundary layer over the bottom of a barge (flat-bottomed boat for carrying freight) varies with the free-stream velocity U, the length of the barge L, the density ρ and the viscosity μ. Find the dimensionless parameters for this problem.

Solution

Worked example 8

University of Nottingham
UK | CHINA | MALAYSIA

Worked example 8

Worked example 9

From 16/17 exam, long question Q17.

The diameter, d, of the dots made by an ink jet printer depend on the ink viscosity, μ, density, ρ, and surface tension, σ, the nozzle diameter, D, the distance, L, of the nozzle to the paper surface, and the ink jet velocity, U. Use dimensional analysis to demonstrate that the set of dimensionless groups (d/D, L/D, Re_D and We_D) can characterize the behaviour of the ink jet.

(Tip: the subscript D in Re_D and We_D represents D is chosen as the characteristic length scale in these dimensionless numbers).

Solution

University of Nottingham
UK | CHINA | MALAYSIA

Worked example 9